Space-Time Finite-Element Exterior Calculus and Variational Discretizations of Gauge Field Theories

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Introduction

Gauge Field Theories

- A **gauge symmetry** is a continuous local transformation on the field variables that leaves the system physically indistinguishable.
- A consequence of this is that the Euler–Lagrange equations are **underdetermined**, i.e., the evolution equations are insufficient to propagate all the fields.
- The **kinematic fields** have no physical significance, but the **dynamic fields** and their conjugate momenta have physical significance.
- The Euler–Lagrange equations are **overdetermined**, and the initial data on a Cauchy surface satisfies a constraint (usually elliptic).
- These degenerate systems are naturally described using **multi-Dirac** mechanics and geometry.

Introduction

Electromagnetism

- Let $\mathbf{E}$ and $\mathbf{B}$ be the electric and magnetic vector fields respectively.
- We can write Maxwell's equations in terms of the scalar and vector potentials ϕ and $\mathbf{A}$ by,

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \nabla^2\phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = 0,$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \Box\mathbf{A} + \nabla\left(\nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t}\right) = 0.$$

- The following transformation leaves the equations invariant,

$$\phi \to \phi - \frac{\partial f}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla f.$$

- The associated Cauchy initial data constraints are,

$$\nabla \cdot \mathbf{B}^{(0)} = 0, \qquad \nabla \cdot \mathbf{E}^{(0)} = 0.$$

Introduction

Gauge conditions

- One often addresses the indeterminacy due to gauge freedom in a field theory through the choice of a **gauge condition**.
- The **Lorenz gauge** is $\nabla \cdot \mathbf{A} = -\frac{\partial \phi}{\partial t}$, which yields,

$$\Box \phi = 0, \qquad \Box \mathbf{A} = 0.$$

- The **Coulomb gauge** is $\nabla \cdot \mathbf{A} = 0$, which yields,

$$\nabla^2 \phi = 0, \qquad \Box \mathbf{A} + \nabla \frac{\partial \phi}{\partial t} = 0.$$

- Given different initial and boundary conditions, some problems may be easier to solve in certain gauges than others. There is no systematic way of deciding which gauge to use for a given problem.

Introduction

Theorem (Noether's Theorem)

- For every continuous symmetry of an action, there exists a quantity that is conserved in time.

Example

- The simplest illustration of the principle comes from classical mechanics: a time-invariant action implies a conservation of the Hamiltonian, which is usually identified with energy.
- More precisely, if $S = \int_{t_a}^{t_b} L(q, \dot{q}) dt$ is invariant under the transformation $t \to t + \epsilon$, then

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = \frac{dH}{dt} = 0$$

Introduction

Theorem (Noether's Theorem for Gauge Field Theories)

- For every differentiable, local symmetry of an action, there exists a **Noether current** obeying a continuity equation. Integrating this current over a spacelike surface yields a conserved quantity called a **Noether charge**.

Example

- The action principle for electromagnetism is $S = \frac{1}{2}\int(\mathbf{B}^2 - \mathbf{E}^2)d^4x$. Applying Noether's theorem to the gauge symmetry yields the following currents:

$$j_0 = \mathbf{E} \cdot \nabla f \qquad\qquad \mathbf{j} = -\mathbf{E}\frac{\partial f}{\partial t} + (\mathbf{B} \times \nabla)f$$

Introduction

Motivation for the approach we take

- Our long-term goal is to develop geometric structure-preserving numerical discretizations that systematically addresses the issue of gauge symmetries. Eventually, we wish to study discretizations of general relativity that address the issue of general covariance.
- Towards this end, we will consider multi-Dirac mechanics based on a Hamilton–Pontryagin variational principle for field theories that is well adapted to degenerate field theories.
- The issue of general covariance also leads us to avoid using a tensor product discretization that presupposes a slicing of spacetime, rather we will consider 4-simplicial complexes in spacetime.
- More generally, we will need to study discretizations that are invariant to some discrete analogue of the gauge symmetry group.

Continuous Hamilton–Pontryagin principle

Pontryagin bundle and Hamilton–Pontryagin principle

- Consider the **Pontryagin bundle** $TQ \oplus T^*Q$, which has local coordinates (q, v, p).
- The **Hamilton–Pontryagin principle** is given by

$$\delta \int [L(q, v) - p(v - \dot{q})] = 0,$$

where we impose the second-order curve condition, $v = \dot{q}$ using Lagrange multipliers p.

Continuous Hamilton–Pontryagin principle

Implicit Lagrangian systems

- Taking variations in q, v, and p yield

$$\begin{aligned}
&\delta \int [L(q,v) - p(v-\dot{q})]dt \\
&\qquad = \int \left[\frac{\partial L}{\partial q}\delta q + \left(\frac{\partial L}{\partial v} - p\right)\delta v - (v-\dot{q})\delta p + p\delta\dot{q}\right] dt \\
&\qquad = \int \left[\left(\frac{\partial L}{\partial q} - \dot{p}\right)\delta q + \left(\frac{\partial L}{\partial v} - p\right)\delta v - (v-\dot{q})\delta p\right] dt,
\end{aligned}$$

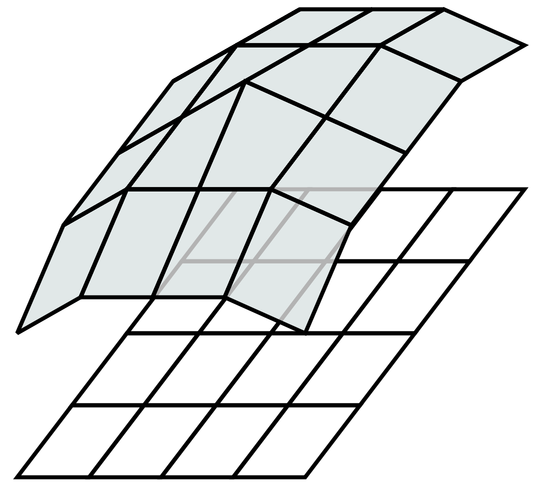
  where we used integration by parts, and the fact that the variation δq vanishes at the endpoints.

- This recovers the **implicit Euler–Lagrange equations**,

$$\dot{p} = \frac{\partial L}{\partial q}, \qquad p = \frac{\partial L}{\partial v}, \qquad v = \dot{q}.$$

Multisymplectic Geometry

Ingredients



- **Base space** $\mathcal{X}$. $(n+1)$-spacetime.
- **Configuration bundle**. Given by $\pi : Y \to \mathcal{X}$, with the fields as the fiber.
- **Configuration** $q : \mathcal{X} \to Y$. Gives the field variables over each spacetime point.
- **First jet** J^1Y. The first partials of the fields with respect to spacetime.

Variational Mechanics

- **Lagrangian density** $L : J^1Y \to \Omega^{n+1}(\mathcal{X})$.
- **Action integral** given by, $\mathcal{S}(q) = \int_{\mathcal{X}} L(j^1q)$.
- **Hamilton's principle** states, $\delta\mathcal{S} = 0$.

Continuous Multi-Dirac Mechanics

Hamilton–Pontryagin for Fields

- In coordinates, the Hamilton–Pontryagin principle for fields is

$$S(y^A, y^A_\mu, p^\mu_A) = \int_U \left[p^\mu_A \left(\frac{\partial y^A}{\partial x^\mu} - v^A_\mu \right) + L(x^\mu, y^A, v^A_\mu) \right] d^{n+1}x.$$

- By taking variations with respect to y^A, v^A_μ and p^μ_A (where δy^A vanishes on ∂U) we obtain the implicit Euler–Lagrange equations,

$$\frac{\partial p^\mu_A}{\partial x^\mu} = \frac{\partial L}{\partial y^A}, \quad p^\mu_A = \frac{\partial L}{\partial v^A_\mu}, \quad \text{and} \quad \frac{\partial y^A}{\partial x^\mu} = v^A_\mu.$$

Covariant Legendre Transform

- The Legendre transform involves both the energy and momentum,

$$p^\mu_A = \frac{\partial L}{\partial v^A_\mu}, \qquad p = L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu.$$

Electromagnetism

Multisymplectic Formulation

- We formulate electromagnetism using the Hamilton–Pontryagin variational principle and the associated multi-Dirac structure.
- The motivation is that the Dirac (and multi-Dirac) formulation is better equipped to handle problems with degenerate Lagrangians, as the implicit Euler–Lagrange equations are in first-order form.
- The electromagnetic potential $A = A_\mu dx^\mu$ is a section of the bundle $Y = T^*X$ of one-forms on spacetime X, where for simplicity, X is $\mathbb{R}^4$ with the Minkowski metric.
- The bundle Y has coordinates (x^μ, A_μ) while J^1Y has coordinates $(x^\mu, A_\mu, A_{\mu,\nu})$.

Electromagnetism

Lagrangian Density

- The electromagnetic Lagrangian density is given by

$$\mathcal{L}(A, j^1 A) = -\frac{1}{4}\mathbf{d}A \wedge \star\mathbf{d}A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ and $\star$ is the Minkowski Hodge star.

Energy-Momentum Tensor

- The Noether quantity associated with space-time covariance is the energy-momentum tensor, which is given by

$$T^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma},$$

and by adding an appropriate total derivative term, we recover the usual energy-momentum tensor,

$$\hat{T}^{\mu\nu} = F^{\mu\lambda}F^\nu_\lambda + \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}.$$

Electromagnetism

Hamilton-Pontryagin Principle

- The Hamilton-Pontryagin action principle is given in coordinates by
$$S = \int_U \left[p^{\mu,\nu} \left(\frac{\partial A_\mu}{\partial x^\nu} - A_{\mu,\nu} \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] d^4x,$$
where U is an open subset of X.

- The implicit Euler–Lagrange equations are given by
$$p^{\mu,\nu} = F^{\mu\nu}, \quad A_{\mu,\nu} = \frac{\partial A_\mu}{\partial x^\nu}, \quad \frac{dp^{\mu,\nu}}{dx^\nu} = 0,$$
and by eliminating $p^{\mu,\nu}$ lead to Maxwell's equations: $\partial_\nu F^{\mu\nu} = 0$.

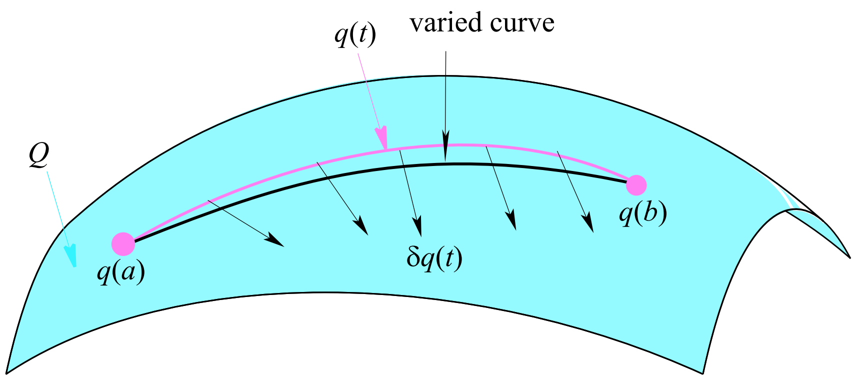
Geometric Discretizations

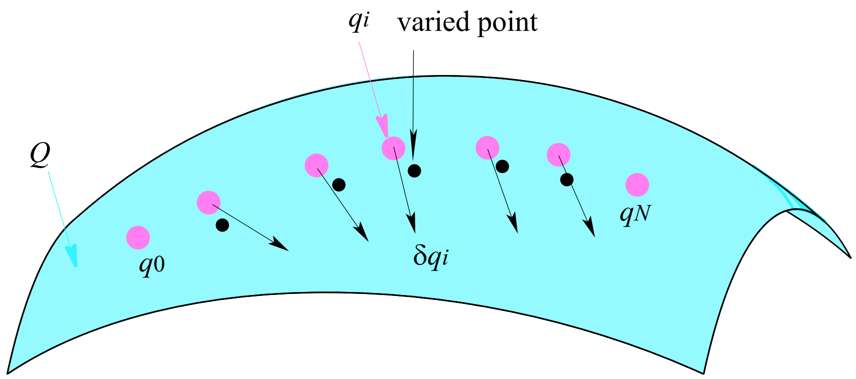
Geometric Integrators

- Given the fundamental role of gauge symmetry and their associated conservation laws in gauge field theories, it is natural to consider discretizations that preserve these properties.
- **Geometric Integrators** are a class of numerical methods that preserve geometric properties, such as symplecticity, momentum maps, and Lie group or homogeneous space structure of the dynamical system to be simulated.
- This tends to result in numerical simulations with better long-time numerical stability, and qualitative agreement with the exact flow.

The Classical Lagrangian View of Variational Integrators

Discrete Variational Principle





- **Discrete Lagrangian**

$$L_d(q_0, q_1) \approx L_d^{\text{exact}}(q_0, q_1) \equiv \int_0^h L\left(q_{0,1}(t), \dot{q}_{0,1}(t)\right) dt,$$

  where $q_{0,1}(t)$ satisfies the Euler–Lagrange equations for L and the boundary conditions $q_{0,1}(0) = q_0$, $q_{0,1}(h) = q_1$.

- This is related to **Jacobi's solution** of the **Hamilton–Jacobi equation**.

The Classical Lagrangian View of Variational Integrators

Discrete Variational Principle

- **Discrete Hamilton's principle**

$$\delta \mathbb{S}_d = \delta \sum L_d(q_k, q_{k+1}) = 0,$$

where q_0, q_N are fixed.

Discrete Euler–Lagrange Equations

- **Discrete Euler-Lagrange equation**

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) = 0.$$

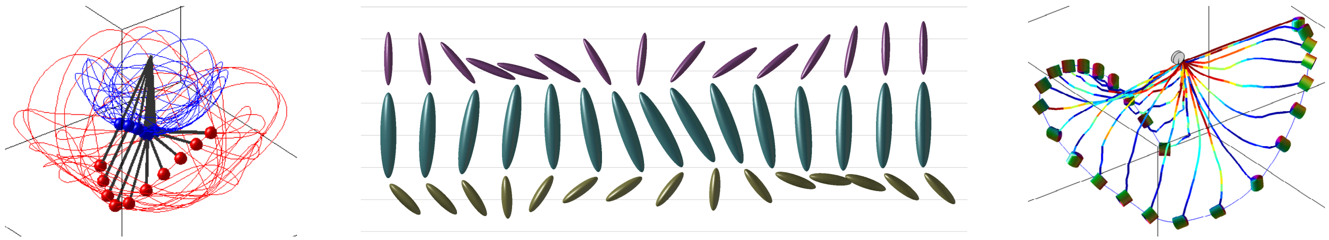
- The associated discrete flow $(q_{k-1}, q_k) \mapsto (q_k, q_{k+1})$ is automatically symplectic, since it is equivalent to,

$$p_k = -D_1 L_d(q_k, q_{k+1}), \quad p_{k+1} = D_2 L_d(q_k, q_{k+1}),$$

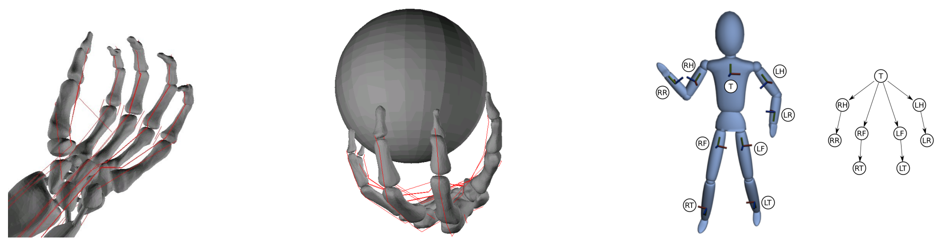
which is the characterization of a symplectic map in terms of a **Type I generating function** (discrete Lagrangian).

Examples of Variational Integrators

• Multibody Systems

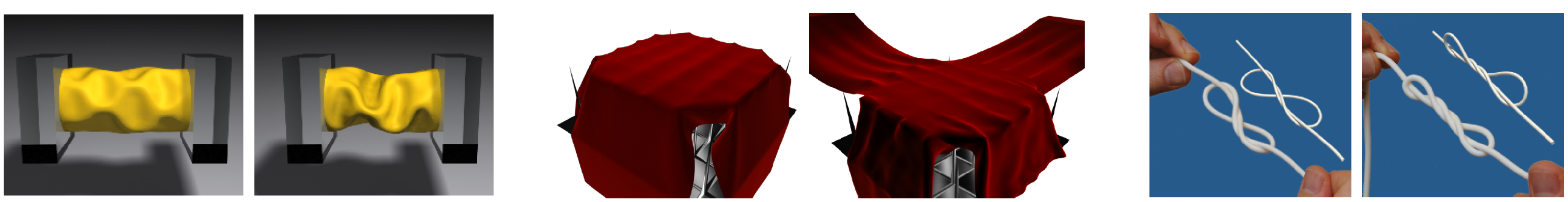


Simulations courtesy of Taeyoung Lee, George Washington University.



Simulations courtesy of Todd Murphey, Northwestern University.

• Continuum Mechanics



Simulations courtesy of Eitan Grinspun, Columbia University.

Lagrangian Variational Integrators

Main Advantages of Variational Integrators

- **Discrete Noether's Theorem**

  If the discrete Lagrangian L_d is (infinitesimally) G-invariant under the diagonal group action on $Q \times Q$,

$$L_d(gq_0, gq_1) = L_d(q_0, q_1)$$

  then the **discrete momentum map** $J_d : Q \times Q \to \mathfrak{g}^*$,

$$\langle J_d\left(q_k, q_{k+1}\right), \xi\rangle \equiv \left\langle D_1 L_d\left(q_k, q_{k+1}\right), \xi_Q\left(q_k\right)\right\rangle$$

  is preserved by the discrete flow.

Lagrangian Variational Integrators

Main Advantages of Variational Integrators

- **Variational Error Analysis**
  Since the exact discrete Lagrangian generates the exact solution of the Euler–Lagrange equation, the exact discrete flow map is *formally* expressible in the setting of variational integrators.
- This is analogous to the situation for B-series methods, where the exact flow can be expressed formally as a B-series.
- If a computable discrete Lagrangian L_d is of order r, i.e.,

$$L_d(q_0, q_1) = L_d^{\text{exact}}(q_0, q_1) + \mathcal{O}(h^{r+1})$$

  then the discrete Euler–Lagrange equations yield an order r accurate symplectic integrator.

Constructing Discrete Lagrangians

Revisiting the Exact Discrete Lagrangian

- Consider an alternative expression for the exact discrete Lagrangian,

$$L_d^{\text{exact}}(q_0, q_1) \equiv \underset{\substack{q\in C^2([0,h],Q)\\ q(0)=q_0, q(h)=q_1}}{\text{ext}} \int_0^h L(q(t), \dot{q}(t))dt,$$

which is more amenable to discretization.

Ritz Discrete Lagrangians

- Replace the infinite-dimensional function space $C^2([0, h], Q)$ with a **finite-dimensional function space**.
- Replace the integral with a **numerical quadrature formula**.

Ritz Variational Integrators

Optimal Rates of Convergence

- A desirable property of a Ritz numerical method based on a finite-dimensional space $F_d \subset F$, is that it should exhibit **optimal rates of convergence**, which is to say that the numerical solution $q_d \in F_d$ and the exact solution $q \in F$ satisfies,

$$\|q - q_d\| \leq c \inf_{\tilde{q} \in F_d} \|q - \tilde{q}\|.$$

- This means that the rate of convergence depends on the best approximation error of the finite-dimensional function space.

Ritz Variational Integrators

Optimality of Ritz Variational Integrators

- Given a sequence of finite-dimensional function spaces $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots \subset C^2([0,h],Q) \equiv \mathcal{C}_\infty$.
- For a correspondingly accurate sequence of quadrature formulas,
$$L_d^i(q_0,q_1) \equiv \underset{q\in\mathcal{C}_i}{\text{ext}}\, h \sum\nolimits_{j=1}^{s_i} b_j^i L(q(c_j^i h), \dot{q}(c_j^i h)),$$
where $L_d^\infty(q_0,q_1) = L_d^{\text{exact}}(q_0,q_1)$.
- Proving $L_d^i(q_0,q_1) \to L_d^\infty(q_0,q_1)$, corresponds to Γ-convergence.
- For optimality, we require the bound,
$$L_d^i(q_0,q_1) = L_d^\infty(q_0,q_1) + c \inf_{\tilde{q}\in\mathcal{C}_i} \|q - \tilde{q}\|,$$
where we need to relate the rate of Γ-convergence with the best approximation properties of the family of approximation spaces.

Ritz Variational Integrators

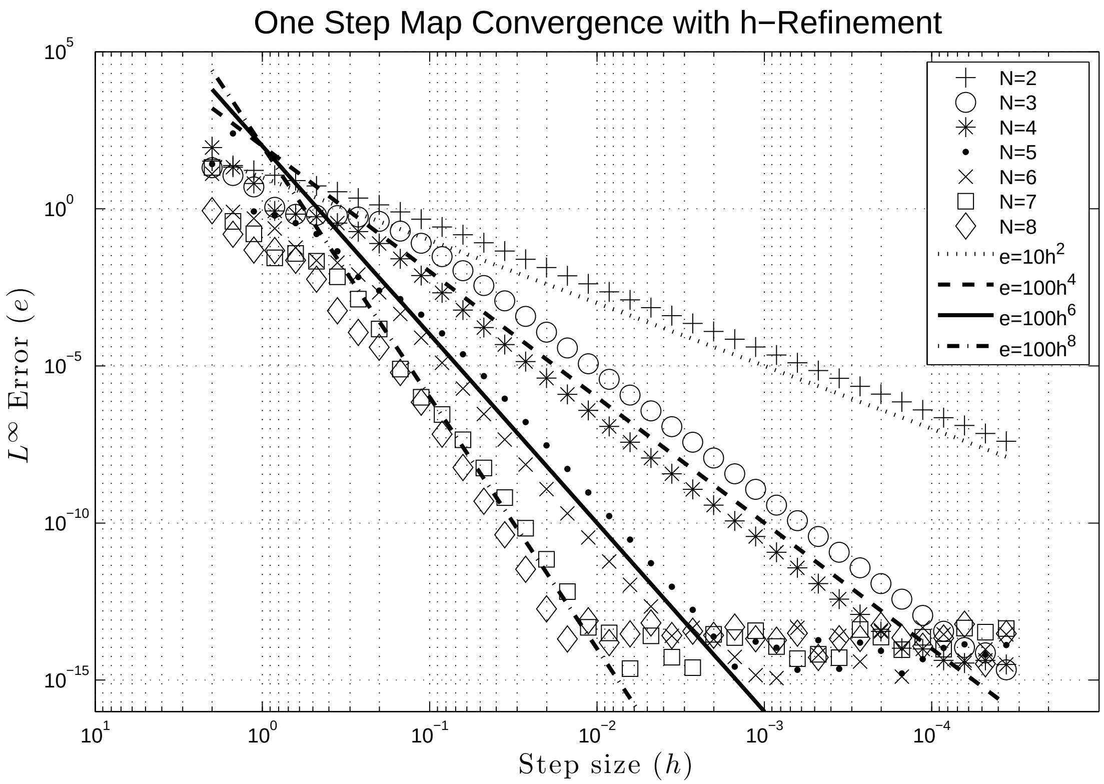
Theorem: Optimality of Ritz Variational Integrators

- Under suitable technical hypotheses:
  - Regularity of L in a closed and bounded neighboorhood;
  - The quadrature rule is sufficiently accurate;
  - The discrete and continuous trajectories *minimize* their actions;

  the Ritz discrete Lagrangian has the same approximation properties as the best approximation error of the approximation space.
- The critical assumption is action minimization. For Lagrangians $L = \dot{q}^T M \dot{q} - V(q)$, and sufficiently small h, this assumption holds.
- Shows that Ritz variational integrators are **order optimal**; spectral variational integrators are **geometrically convergent**.

Ritz Variational Integrators

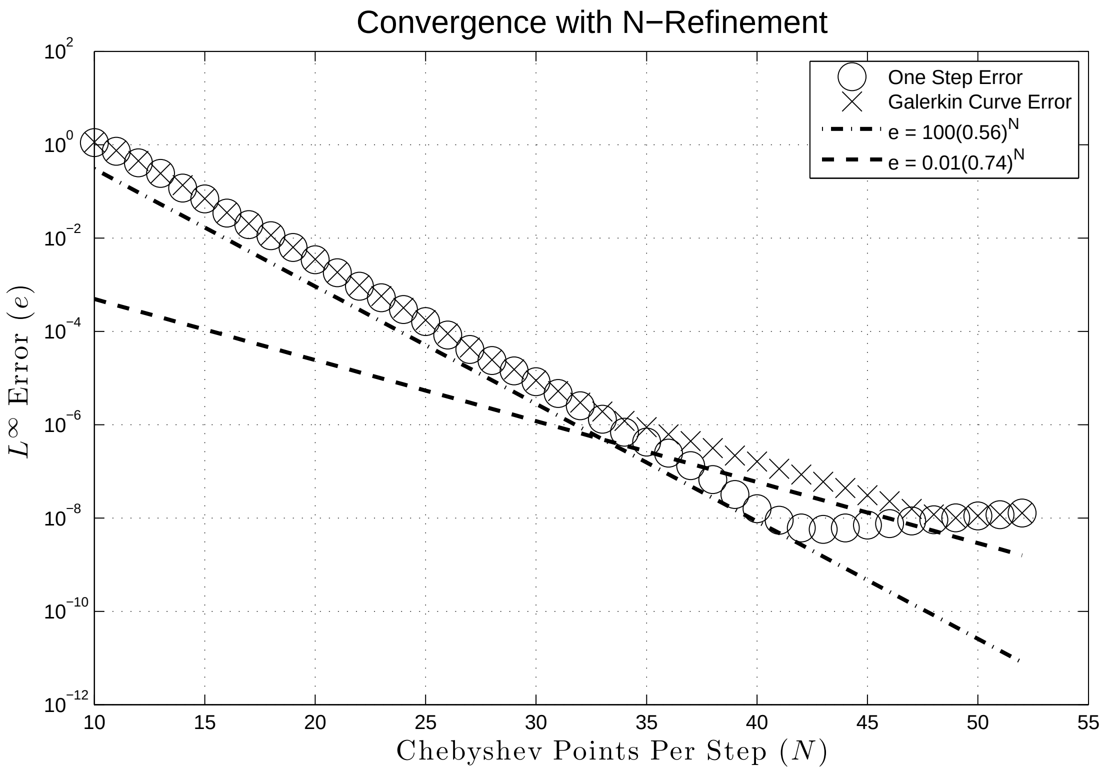
Numerical Results: Order Optimal Convergence



- Order optimal convergence of the Kepler 2-body problem with eccentricity 0.6 over 100 steps of $h = 2.0$.

Spectral Ritz Variational Integrators

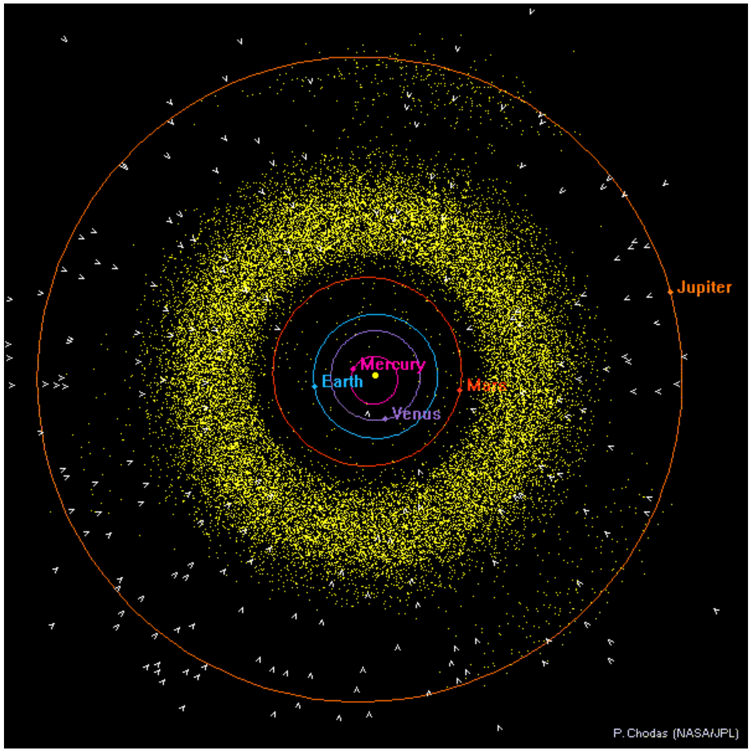
Numerical Results: Geometric Convergence

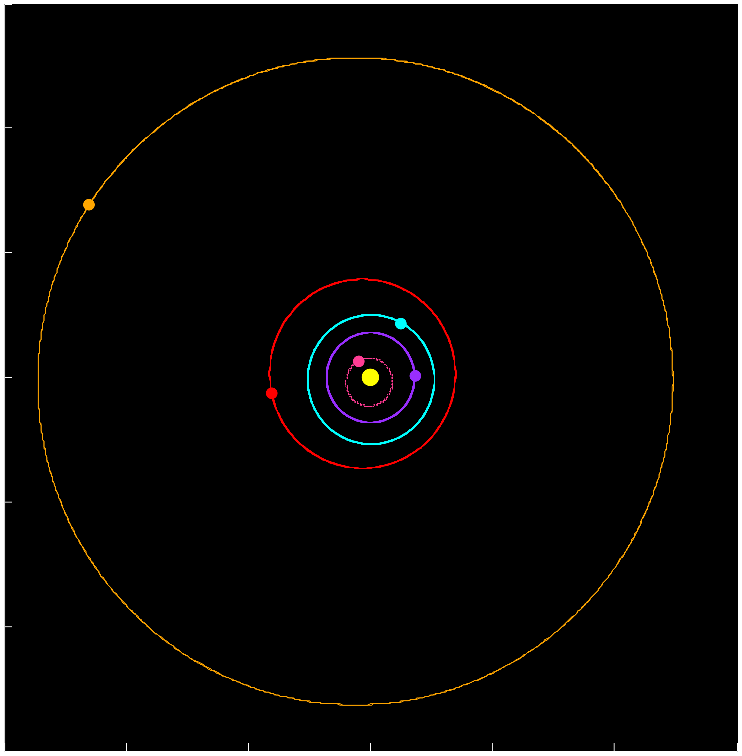


- Geometric convergence of the Kepler 2-body problem with eccentricity 0.6 over 100 steps of $h = 2.0$.

Spectral Ritz Variational Integrators

Numerical Experiments: Solar System Simulation

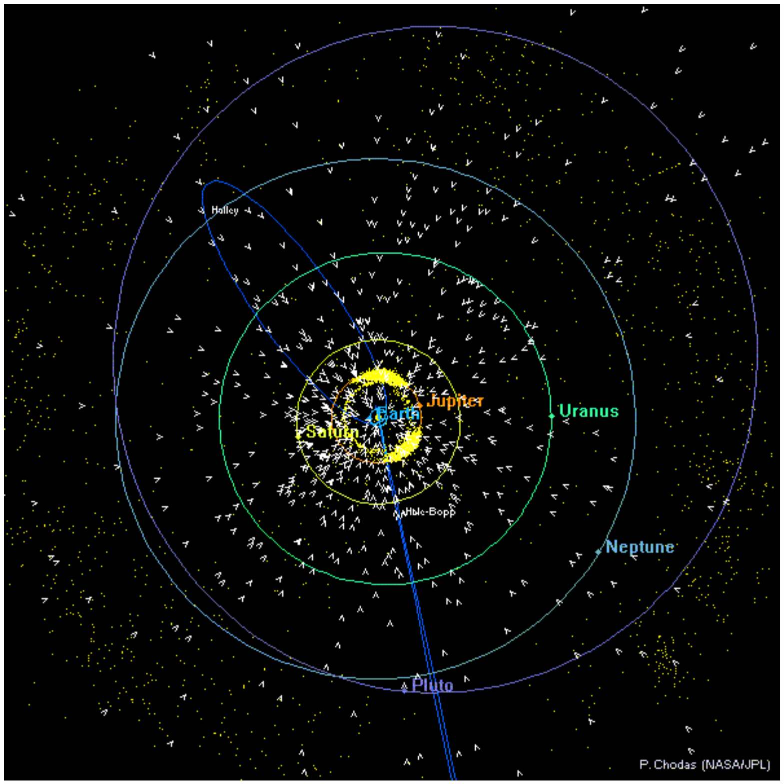


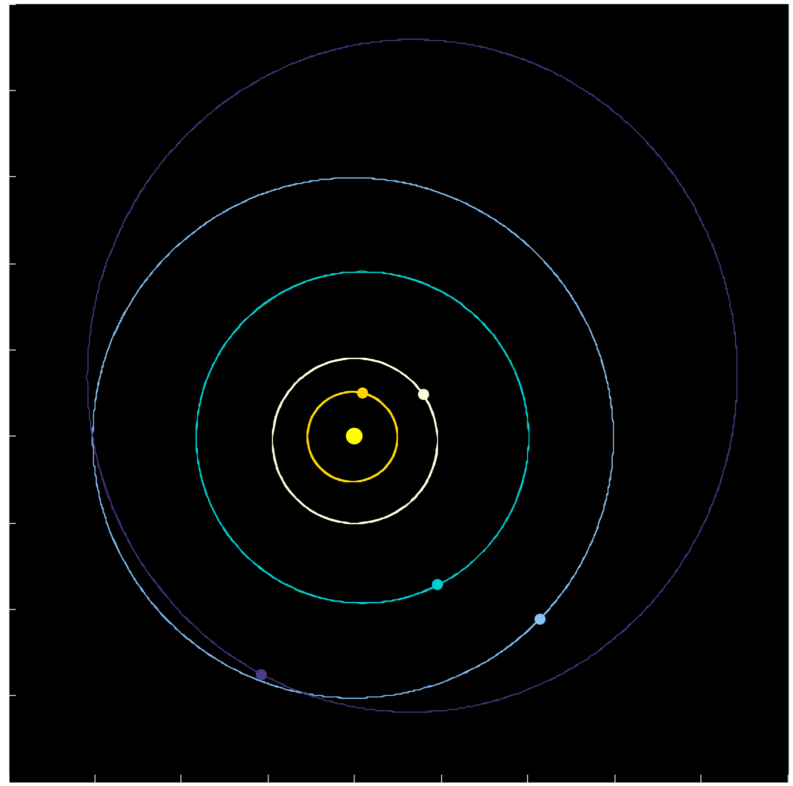


- Comparison of inner solar system orbital diagrams from a spectral variational integrator and the JPL Solar System Dynamics Group.
- $h = 100$ days, $T = 27$ years, 25 Chebyshev points per step.

Spectral Ritz Variational Integrators

Numerical Experiments: Solar System Simulation

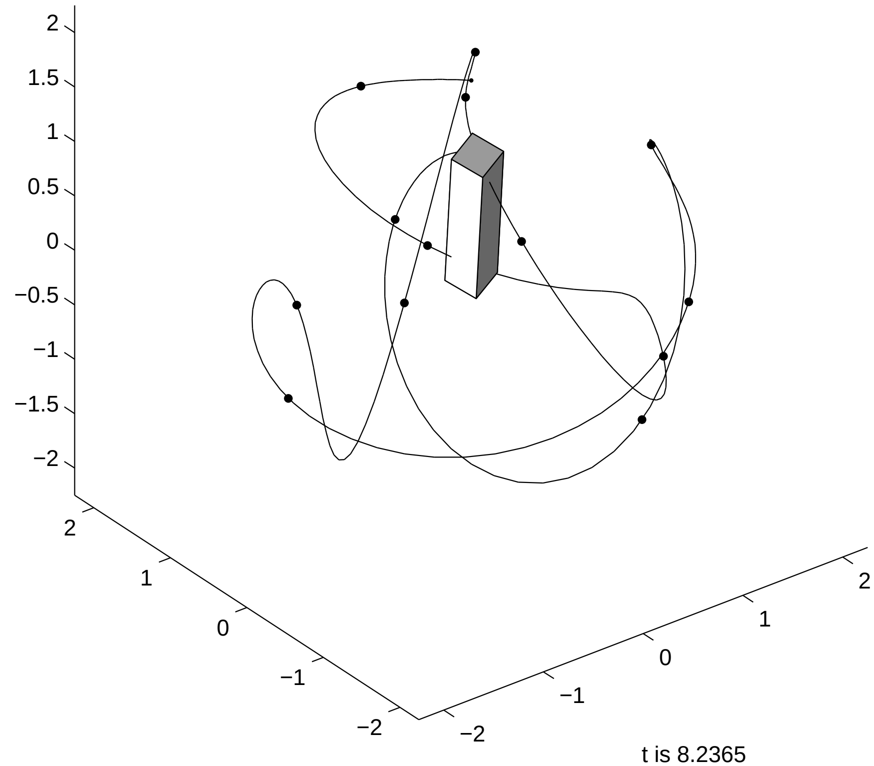




- Comparison of outer solar system orbital diagrams from a spectral variational integrator and the JPL Solar System Dynamics Group. Inner solar system was aggregated, and $h = 1825$ days.

Spectral Lie Group Variational Integrators

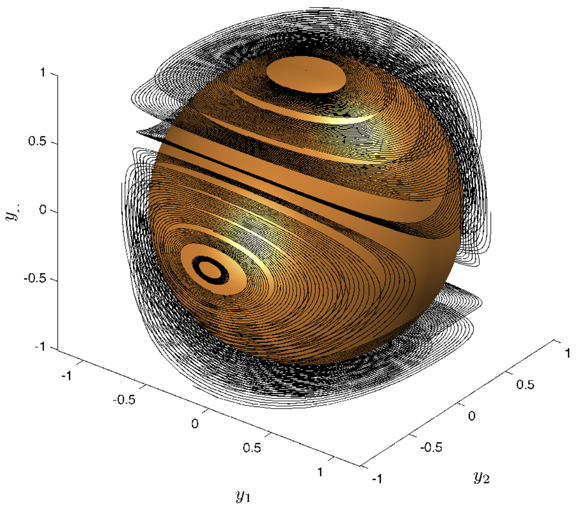
Numerical Experiments: 3D Pendulum



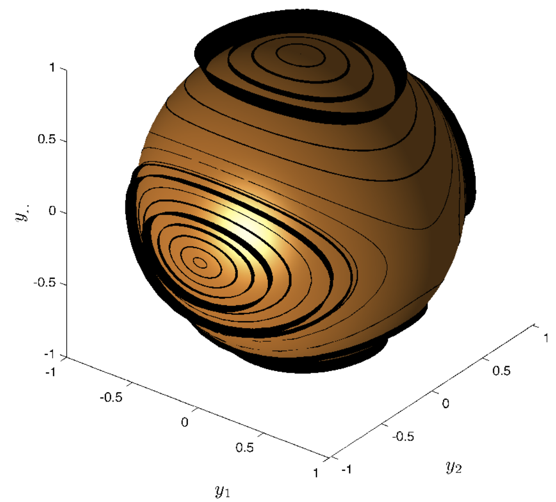
- $n = 20$, $h = 0.6$. The black dots represent the discrete solution, and the solid lines are the Ritz curves. Some steps involve a rotation angle of almost π, which is close to the chart singularity.

Spectral Lie Group Variational Integrators

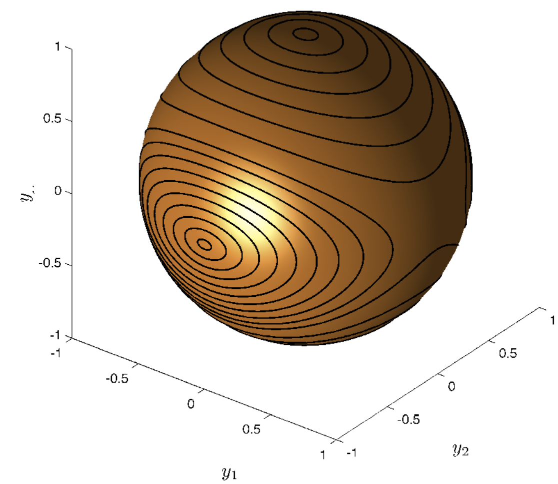
Numerical Experiments: Free Rigid Body



Explicit Euler



MATLAB ode45

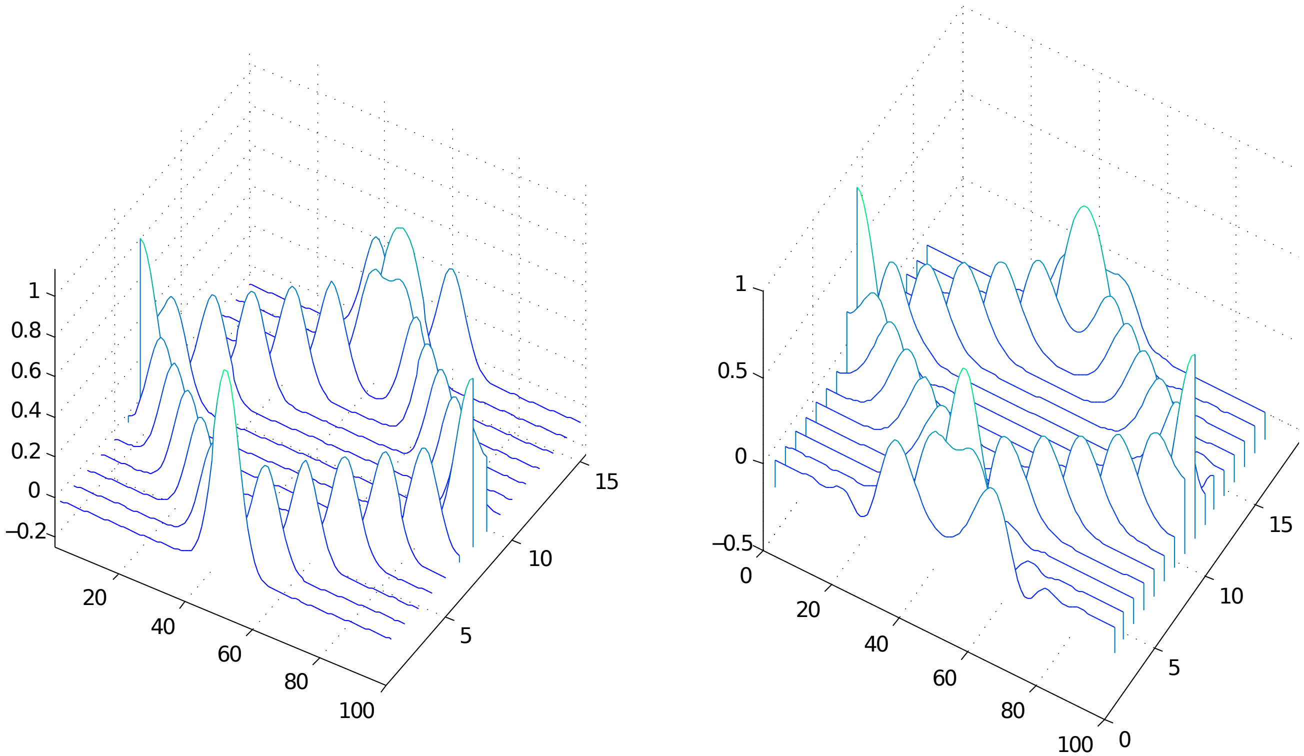


Lie Group Variational Integrator

- The conserved quantities are the norm of body angular momentum, and the energy. Trajectories lie on the intersection of the angular momentum sphere and the energy ellipsoid.
- These figures illustrate the extent to the numerical methods preserve the quadratic invariants.

Spectral Variational Integrators

Numerical Experiments: Spectral Wave Equation



- The wave equation $u_{tt} = u_{xx}$ on S^1 is described by the Lagrangian density function, $L(\varphi, \dot{\varphi}) = \frac{1}{2}\left|\dot{\varphi}(x,t)\right|^2 - \frac{1}{2}\left|\nabla\varphi(x,t)\right|^2.$
- Discretized using spectral in space, and linear in time.

Multisymplectic Exact Discrete Lagrangian

■ What is the PDE analogue of a generating function?

- Recall the implicit characterization of a symplectic map in terms of generating functions:

$$\begin{cases} p_k = -D_1 L_d(q_k, q_{k+1}) \\ p_{k+1} = D_2 L_d(q_k, q_{k+1}) \end{cases} \qquad \begin{cases} p_k = D_1 H_d^+(q_k, p_{k+1}) \\ q_{k+1} = D_2 H_d^+(q_k, p_{k+1}) \end{cases}$$

- Symplecticity follows as a trivial consequence of these equations, together with $\mathbf{d}^2 = 0$, as the following calculation shows:

$$\begin{aligned} \mathbf{d}^2 L_d(q_k, q_{k+1}) &= \mathbf{d}(D_1 L_d(q_k, q_{k+1}) dq_k + D_2 L_d(q_k, q_{k+1}) dq_{k+1}) \\ &= \mathbf{d}(-p_k dq_k + p_{k+1} dq_{k+1}) \\ &= -dp_k \wedge dq_k + dp_{k+1} \wedge dq_{k+1} \end{aligned}$$

Multisymplectic Exact Discrete Lagrangian

Analogy with the ODE case

- We consider a multisymplectic analogue of Jacobi's solution:

$$L_d^{\text{exact}}(q_0, q_1) \equiv \int_0^h L\left(q_{0,1}(t), \dot{q}_{0,1}(t)\right) dt,$$

  where $q_{0,1}(t)$ satisfies the Euler–Lagrange boundary-value problem.

- This is given by,

$$L_d^{\text{exact}}(\varphi|_{\partial\Omega}) \equiv \int_\Omega L(j^1\tilde{\varphi})$$

  where $\tilde{\varphi}$ satisfies the boundary conditions $\tilde{\varphi}|_{\partial\Omega} = \varphi|_{\partial\Omega}$, and $\tilde{\varphi}$ satisfies the Euler–Lagrange equation in the interior of Ω.

Multisymplectic Exact Discrete Lagrangian

Multisymplectic Relation

- If one takes variations of the **multisymplectic exact discrete Lagrangian** with respect to the boundary conditions, we obtain,

$$\partial_{\varphi(x,t)} L_d^{\text{exact}}(\varphi|_{\partial\Omega}) = p_\perp(x,t),$$

where $(x,t) \in \partial\Omega$, and $p_\perp$ is a codimension-1 differential form, that by Hodge duality can be viewed as the normal component (to the boundary $\partial\Omega$) of the multimomentum at the point (x,t).

- These equations, taken at every point on $\partial\Omega$ constitute a **multisymplectic relation**, which is the PDE analogue of,

$$\begin{cases} p_k = -D_1 L_d(q_k, q_{k+1}) \\ p_{k+1} = D_2 L_d(q_k, q_{k+1}) \end{cases}$$

where the sign comes from the orientation of the boundary.

Gauge Symmetries and Variational Discretizations

Theorem (Noether's Theorem)

- For every continuous symmetry of an action, there exists a quantity that is conserved in time.

Theorem (Noether's Theorem for Gauge Field Theories)

- For every differentiable, local symmetry of an action, there exists a **Noether current** obeying a continuity equation. Integrating this current over a spacelike surface yields a conserved quantity called a **Noether charge**.

Implications for Geometric Integration

- Since gauge symmetries are associated with conserved quantities, we need finite-elements that are (approximately) group-equivariant.
- Two current approaches, **spacetime finite-element exterior calculus**, and **geodesic finite-elements**.

Whitney Forms

Barycentric Coordinates

- The **Whitney k-forms** are dual to k-simplices via integration.
- Described in Whitney's "Geometric Integration Theory" (1957).
- The **barycentric coordinates** λ_i of an n-simplex with vertex vectors $v_0, v_1, \ldots v_n$ are functions of the position vector x such that

$$\sum_{i=0}^{n} \lambda_i v_i = x \qquad \sum_{i=0}^{n} \lambda_i = 1$$

Whitney Forms in terms of Barycentric Coordinate

- Let $\rho := [v_0, v_1, \ldots v_j]$. The **Whitney j-form** ${}^j w_\rho$ is:

$${}^j w_\rho = j! \sum_{i=0}^{j} (-1)^i \lambda_i d\lambda_0 \wedge d\lambda_1 \wedge \ldots \widehat{d\lambda_i} \wedge \cdots \wedge d\lambda_j$$

The λ_i are the barycentric coordinates, and the hat indicates an omitted term.

Whitney Forms

Extensions to Flat Pseudo-Riemannian Manifolds

- There have been many approaches based on a space-time splitting, including tensor product finite-element exterior calculus (Arnold, Boffi, Bonizzoni 2013), cubical FEEC (Arnold, Awanou 2012), prismal discrete exterior calculus for electromagnetism (Desbrun, Hirani, ML, Marsden 2005), and their generalization to asynchronous variational integrators (Stern, Tong, Desbrun, Marsden 2008).
- However, we wish to consider a space-time discretization, without a slicing of spacetime, in a FEEC setting.
- While Whitney forms, and the higher-order FEEC generalizations are represented in terms of barycentric coordinates, the Hodge star is most naturally expressed in terms of space-time adapted coordinates.

Whitney Forms

Key Properties

- Linearity in x, the position vector. Comes from the fact that the barycentric coordinate map is linear.
- Antisymmetric with respect to vertex interchange.
- Whitney forms vanish on the complementary subsimplex.
- Normalization: $\int_\rho {}^j w_\rho = 1$.

Generalization to Spacetime

- The four conditions above are sufficient to reconstruct forms completely equivalent to Whitney's original barycentric formulation.

Whitney Forms

Theorem (Spacetime Whitney Forms)

- Let $\sigma := [v_0, v_1, ...v_n]$, an ordered set of vertex vectors, represent an oriented n-simplex on a flat n-dimensional manifold. Let $\rho \subseteq \sigma$ be a j-subsimplex, and $\tau = \sigma \setminus \rho$ be the ordered complement of ρ in σ. The Whitney j-form over ρ can be written as

$$^{j}w_{\rho} = \frac{sgn(\rho \cup \tau)}{\star vol(\sigma)} \frac{j!}{n!} \left(\star \bigwedge_{v_k \in \tau} (v_k - x)^{\flat} \right),$$

with $vol(\sigma) = \frac{1}{n!} \bigwedge_{i=1}^{n} (v_i - v_0)^{\flat}$, the volume form of σ.

Theorem (Vector Proxy Spacetime Whitney Forms)

$$^{j}w_{\rho}[W_j] = \star\star sgn(\rho \cup \tau) \frac{j!}{n!} \frac{\left\langle \bigwedge_{i=1}^{n} (v_i - v_0), \left(\bigwedge_{v_k \in \tau} (v_k - x)\right) \wedge W_j \right\rangle}{\left\langle \bigwedge_{i=1}^{n} (v_i - v_0), \bigwedge_{i=1}^{n} (v_i - v_0) \right\rangle}$$

Hodge Dual of Whitney Forms

Properties

- Our formulation also provides the first concrete characterization of the dual Whitney forms $\star^j w_\rho$.
- From our formula, we can make the following observations about the Hodge dual of Whitney forms:
  - They do not lie in the span of ${}^j w_\rho$.
  - $d(\star^j w_\rho) = 0$, i.e. they are closed.
  - Their support are not naturally associated with σ and its sub-simplices.
  - Work done by Harrison (2006) suggests that they naturally live on the geometrical dual $\star\sigma$ and its appropriately dualized sub-simplices.

Applications to Gauge Field Theories

Gauge Freedom in Electromagnetism

- We consider electromagnetism as a gauge field theory.
- In Minkowski spacetime, the action for EM can be written as,

$$S = \frac{1}{2}\int_M F \wedge \star F,$$

where $F = dA$ and $A = (\phi, \mathbf{A})^\flat$, the 4-potential. The gauge symmetry is $A \to A + df$, where f is a 0-form.

- Maxwell's equations can be written as,

$$\delta F = 0, \qquad\qquad dF = 0.$$

- The Noether current associated with the gauge symmetry is,

$$j = \star((\star F) \wedge df).$$

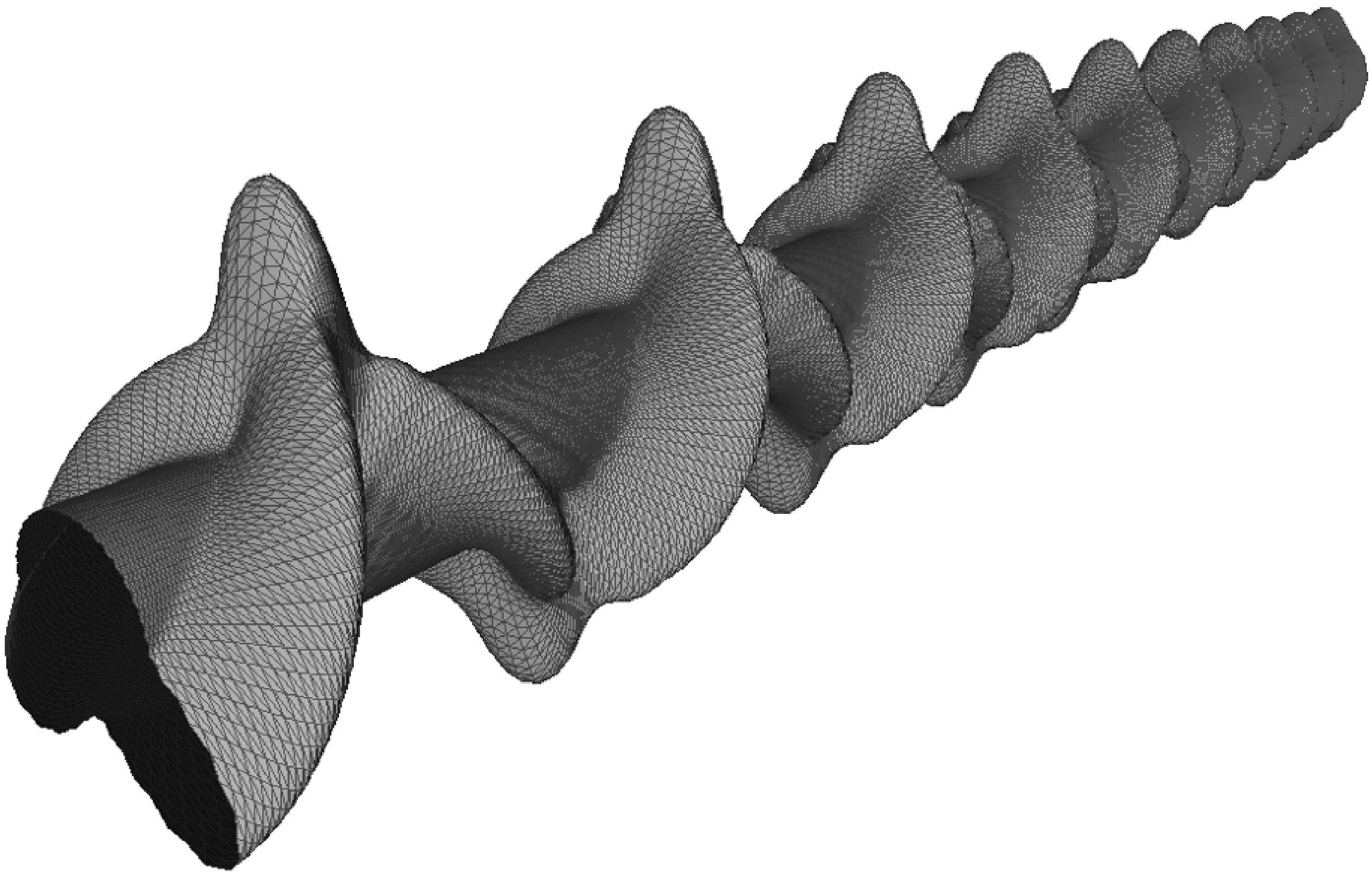
Applications to Gauge Field Theories

Gauge Freedom in Electromagnetism

- We now apply our spacetime FEEC discretization. Take $A = \sum_{i<j} a_{ij} w_{ij}$, and $f = \sum_i b_i w_i = \sum_i b_i \lambda_i$. Then,
$$A' = A + df = \sum_{i<j} (a_{ij} + b_j - b_i) w_{ij}.$$
- So the spacetime Whitney forms automatically satisfy the Lorenz gauge, $\delta A = 0$. Going one step further,
$$dA' = \sum_{i<j} \sum_k a_{ij} w_{ijk} = dA.$$
- Thus, our spacetime FEEC framework automatically satisfies gauge invariance, and in particular, $dF = 0$ is automatically satisfied. Therefore,
$$\int_{\sum \rho} j = \int_{\sum \rho} \star (F \wedge df) = 0.$$

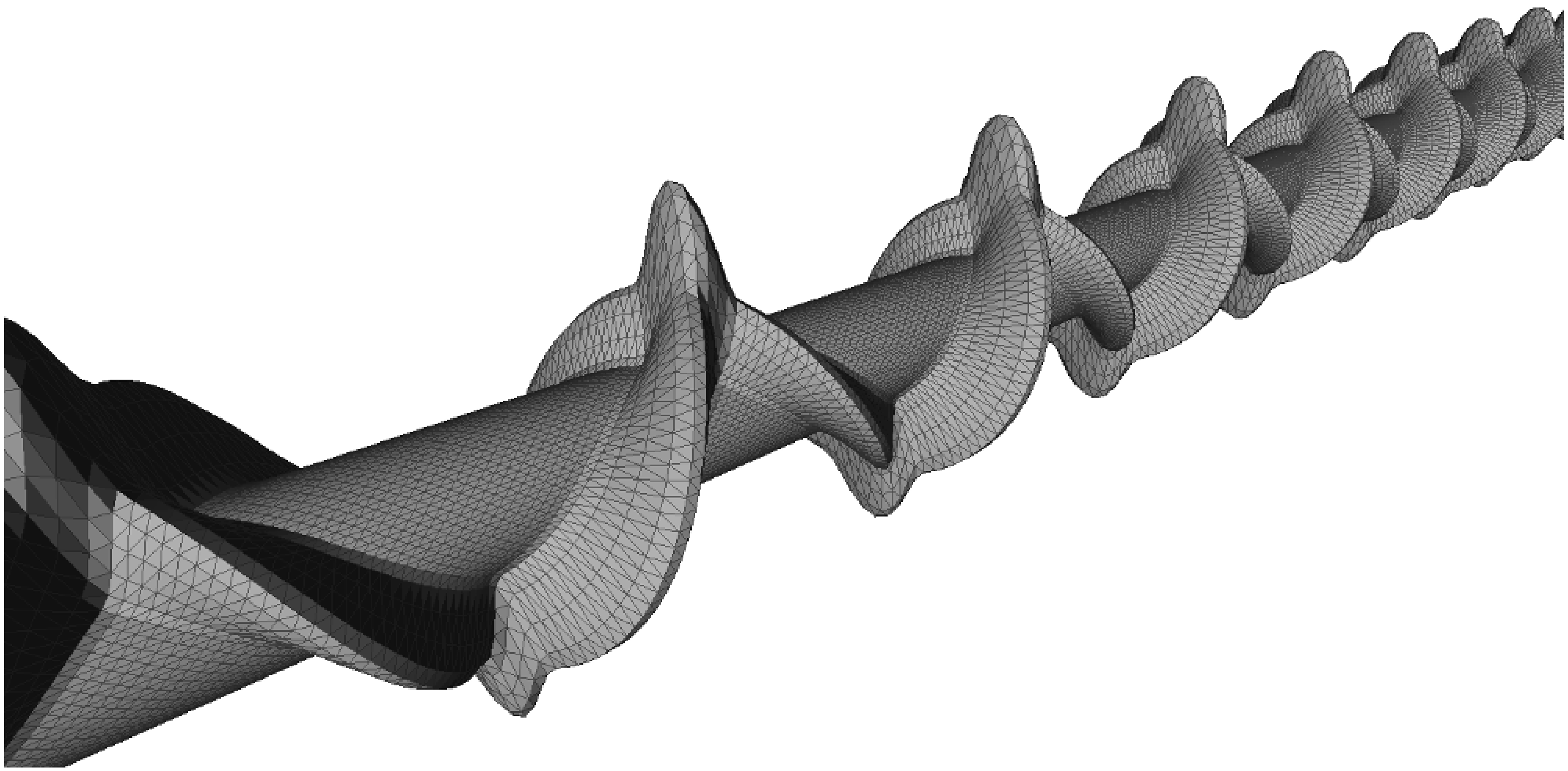
Spacetime Whitney Forms for the Wave-Equation on $S^1 \times \mathbb{R}$

- **Unstructured spacetime mesh**



Spacetime Whitney Forms for the Wave-Equation on $S^1 \times \mathbb{R}$

- **Spacetime mesh align along characteristics**



Lorentzian Metric-Valued Geodesic Finite-Elements

Geodesic Finite-Elements

- On a Riemannian manifold (M, g), the **geodesic finite-element** $\varphi : \Delta^n \to M$ associated with a set of linear space finite-elements $\{v_i : \Delta^n \to \mathbb{R}\}_{i=0}^n$ is given by the **Fréchet** (or **Karcher**) mean,

$$\varphi(x) = \arg\min_{p \in M} \sum\nolimits_{i=0}^{n} v_i(x)(\mathrm{dist}(p, m_i))^2,$$

  where the optimization problem involved can be solved using optimization algorithms developed for matrix manifolds.

- The spatial derivatives of the geodesic finite-element can be computed in terms of an associated optimization problem.

- The advantage is that geodesic finite-elements inherit the approximation properties of the underlying linear space finite-element.

Lorentzian Metric-Valued Geodesic Finite-Elements

Applied to Lorentzian Metric Valued Functions

- Endow the space of Lorentzian metrics with a Riemannian metric, which is achieved by realizing the space of Lorentzian metrics as a symmetric space $GL_4(\mathbb{R})/O_{1,3}$.
- $O_{1,3}$ acts on $GL_4(\mathbb{R})$ by conjugation, and the involution $s : \mathfrak{gl}_4 \to \mathfrak{gl}_4$ is given by $s(v) = -bv^T b$, where $b = \mathrm{diag}(-1, 1, 1, 1)$.
- Then, the Riemannian metric on $GL_4(\mathbb{R})$ induces a well-defined Riemannian metric on the space of Lorentzian metrics.
- The construction of higher-order higher-regularity GFEMs require the construction of relaxations of partitions of unity, in that one needs non-negative shape functions, such that the sum is nonzero everywhere, but one does not require that the sum is normalized.

Summary

- Gauge field theories exhibit gauge symmetries that impose Cauchy initial value constraints, and are also underdetermined.
- These result in degenerate field theories that can be described using multi-Dirac mechanics and multi-Dirac structures.
- Described a systematic framework for constructing and analyzing Ritz variational integrators, and the extension to Hamiltonian PDEs.
- Generalized Whitney forms to spacetime on flat pseudo-Riemannian manifolds, and provided an explicit characterization for the Hodge dual of Whitney forms.
- Spacetime Whitney forms provide a method for preserving the gauge symmetry of electromagnetism at a discrete level.

References

1. J. Hall, ML, Spectral Variational Integrators, Numerische Mathematik, Online First, 2014.
2. ML, T. Ohsawa, Variational and Geometric Structures of Discrete Dirac Mechanics, Found. Comput. Math., 11(5), 529–562, 2011.
3. J. Vankerschaver, C. Liao, ML, Generating Functionals and Lagrangian PDEs, J. Math. Phys., 54(8), 082901, 2013.
4. J. Vankerschaver, H. Yoshimura, ML, The Hamilton-Pontryagin Principle and Multi-Dirac Structures for Classical Field Theories, J. Math. Phys., 53(7), 072903, 2012.
5. J. Vankerschaver, H. Yoshimura, ML, On the Geometry of Multi-Dirac Structures and Gerstenhaber Algebras, J. Geom. Phys., 61(8), 1415–1425, 2011.
6. J. Salamon, J. Moody, ML, Geometric Representations of Whitney Forms and their Generalization to Minkowski Spacetime, under revision, 2015.